

Willis and Kerswell Reply: In our letter [1] it was reported that in pipe flow the median time τ for relaminarisation of localised turbulent disturbances closely follows the scaling $\tau \sim 1/(Re_c - Re)$. This conclusion was based on data from collections of 40 to 60 independent simulations at each of six different Reynolds numbers, Re . In the previous comment, Hof *et al.* estimate τ differently for the point at lowest Re . Although this point is the most uncertain, it forms the basis for their assertion that the data might then fit an exponential scaling $\tau \sim \exp(A Re)$, supporting [2], for some constant A . The most certain point (at largest Re) does not fit their conclusion and is rejected. We clarify why their argument for rejecting this point is flawed. The median τ is estimated from the distribution of observations, and it is shown that the correct part of the distribution is used. The data is sufficiently well determined to show that the exponential scaling cannot be fit to the data over this range of Re , whereas the $\tau \sim 1/(Re_c - Re)$ fit is excellent, indicating critical behaviour and supporting experiments [3].

The data median was used in our original analysis as the estimator for the parameter of the exponential distribution. Estimating via the log-probability plot as in [2], the range of possible lines that can be drawn does not at all reflect the true error in the mean. In the following we estimate confidence intervals by using the bootstrapping method [4] applied to the maximum likelihood estimator of $1/\tau$. Any truncated data is integrated in a logical fashion into this resampling method by further resampling.

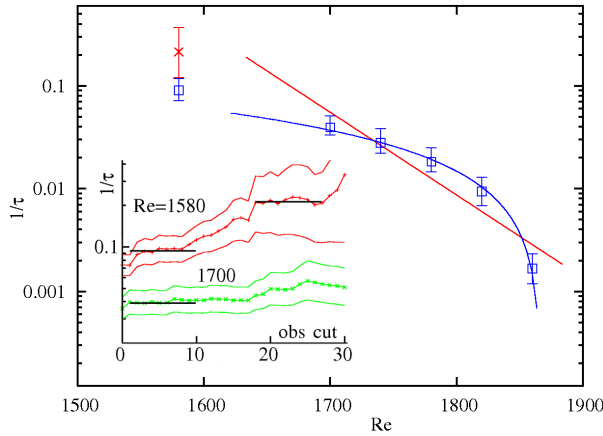


FIG. 1: Data with 95% confidence intervals. Inset: median and 95% bracketing lines *vs.* number of observations removed.

The data are affected by a transient time t_0 for adjustment from the initial condition at $Re = 1900$ to the operating Re . This is of around $50 D/U$ in the worst case. In the inset to Fig. 1 we calculate $1/\tau$ with 95% confidence intervals as a function of the number of smallest observations ignored. In the original analysis a maximum of 10% of the data was removed and for $Re = 1580$ gives the esti-

mate indicated by the first solid bar. Cutting more leads to a rise in $1/\tau$ but the uncertainty becomes very large. A possible time as suggested by Hof *et al.* is $\tau = 4.9 D/U$ (second higher bar, and red \times on Re plot), but it is an order of magnitude smaller than t_0 . As $\tau \ll t_0$ so few observations survive past the transient that nearly 20 of 40 observations must be cut. Aside from statistical uncertainty, this short time is far from the asymptotic $\tau \rightarrow \infty$ and it is questionable that a self-sustaining mechanism is really involved. The exponential fit of Hof *et al.*, however, relies heavily on this data point.

The largest step in Re is from 1580 to 1700. Here the adjustment in Re from the initial condition is smaller, the transient time is shorter, the typical observation time is longer and the ‘tail’ exponential distribution is clearly seen. It is this part of the distribution which has been fit – after the first few points the estimate for $1/\tau$ (solid bar) is within the confidence interval even when large proportions of the data are ignored.

At $Re = 1860$ Hof *et al.* state that the data is inconclusive because the probability $P(T)$ is only shown for $T < 1000 D/U$ [5]. A maximum observation time is not a limitation provided that the probability distribution observed within this time reflects that beyond it. The data fit a straight line on the log-plot and therefore very well the exponential distribution, i.e. the process is memoryless. For the behaviour to change to a second memoryless process beyond $T = 1000 D/U$ is contradictory, requiring some property of the flow to be preserved to such long times.

In Fig. 1 the best-fit exponential and inverse scalings are fit to data at the 5 largest Re . It is evident that most points are significantly different at the 5% level from the exponential scaling over this range of Re , no matter how it is fit. In contrast $1/(Re_c - Re)$ with simple power -1 is a very natural fit.

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A. P. Willis and R. R. Kerswell,
Department of Mathematics,
University of Bristol, BS8 1TW, U. K.

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 - [2] B. Hof, J. Westerweel, T. Schneider, and B. Eckhardt, Nature **443**, 59 (2006).
 - [3] J. Peixinho and T. Mullin, Phys. Rev. Lett. **96**, 094501 (2006).
 - [4] B. Efron, Ann. Stat. **7**, 1 (1979).
 - [5] In the supplementary to their article [2] the numerical data (Fig. 2) is shown for $T < 3000 R/(2U) = 750 D/U$, being less than our own limit.